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THE EXTENT OF STRENGTH IN THE CLUB FILTERS

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ABSTRACT

In this paper we consider whether the minimal normal filter on $\mathcal{P}_\kappa\lambda$, the club filter, can have strong properties like saturation, pre-saturation, or cardinal preserving. We prove in a number of cases that the answer is no. In the case of saturation, Foreman and Magidor prove the answer is always no (except in the case $\kappa = \lambda = \aleph_1$—and in this case saturation is known to be consistent).

1. Introduction

This paper gives a number of partial results towards the following conjectures. Unless otherwise noted, κ is a regular, uncountable cardinal and λ is an infinite cardinal ($\lambda \geq \kappa$).

CONJECTURE 1: *The club filter on $\mathcal{P}_\kappa\lambda$ is not precipitous — unless λ is regular.*

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CONJECTURE 2: *The club filter on $\mathcal{P}_\kappa\lambda$ is not pre-saturated — unless $\kappa = \aleph_1$ and λ is regular or $\kappa = \lambda$ is weakly inaccessible.*

The corresponding conjecture for saturation has been established by Foreman and Magidor ([FM]):

THEOREM (Foreman-Magidor): *The club filter on $\mathcal{P}_\kappa\lambda$ is not saturated — unless $\kappa = \lambda = \aleph_1$.*

The results of section 2 of this paper are the authors' partial results towards the above theorem. Shortly after the results of this paper were announced, Foreman and Magidor proved the above theorem. Their paper gives a self-contained proof of the theorem. Also, in the case covered by Theorem 2.10, they establish the stronger result that the club filter is not even λ^{++} saturated.

Remarks:

1. [She87] It is consistent that the club filter on $\aleph_1$ is saturated (assuming the consistency of a Woodin cardinal).
2. [Git95] It is consistent that the club filter on κ, κ weakly inaccessible, is pre-saturated (assuming the consistency of an up-repeat point).
3. [Gol92] If δ is Woodin then, for every regular λ ($\aleph_1 \leq \lambda < \delta$),
$$V^{\mathrm{Col}(\lambda,<\delta)} \models \text{"the club filter on } \mathcal{P}_{\aleph_1}\lambda \text{ is pre-saturated"}.$$
4. [Gol] If δ is Woodin then, for every regular $\kappa < \lambda$ ($\aleph_1 \leq \kappa \leq \lambda < \delta$),
$$V^{\mathrm{Col}(\lambda,<\delta)} \models \text{"the club filter on } \mathcal{P}_{\kappa}\lambda \text{ is precipitous"}.$$

We now give our basic definitions and conventions.

$\mathcal{F}$ is a **normal filter** on $\mathcal{P}(\lambda)$ if

1. $\mathcal{F} \subseteq \mathcal{PP}(\lambda)$ is a filter.
2. (fine) $\forall\alpha \in \lambda\ \{a \subseteq \lambda \mid \alpha \in a\} \in \mathcal{F}$.
3. (normal) If $C_\alpha \in \mathcal{F}$ ($\alpha \in \lambda$), then $\{a \subseteq \lambda \mid \forall\alpha \in a(a \in C_\alpha)\} \in \mathcal{F}$.

Throughout this paper, *filter* will mean normal filter.

$$\mathcal{F}^+ =_{\mathrm{def}} \{A \subseteq \mathcal{P}(\lambda) \mid \forall C \in \mathcal{F}\ (C \cap A \neq \varnothing)\}.$$

$\mathcal{F}^+$ has an associated partial ordering: $A \leq B$ iff $A \subseteq B$.

A filter $\mathcal{F}$ on $\mathcal{P}(\lambda)$ is **saturated** if every antichain in $\mathcal{F}^+$ has size $\leq \lambda$. $\mathcal{F}$ is **pre-saturated** if, given antichains $\mathcal{A}_\alpha$ ($\alpha < \lambda$) and $S \in \mathcal{F}^+$, there is a $T \leq S$ such that, for all $\alpha < \lambda$, $|\{A \in \mathcal{A}_\alpha \mid A \cap T \in \mathcal{F}^+\}| \leq \lambda$.

Forcing with $\mathcal{F}^+$ extends $\mathcal{F}$ to a V-normal, V-ultrafilter $\mathcal{G}$—so we get a generic embedding $j\colon V \to \mathrm{Ult}(V,\mathcal{G}) \subseteq V[\mathcal{G}]$.

$\mathcal{F}$ is **precipitous** if this ultrapower is always well-founded. If $\mathcal{F}$ is pre-saturated, then $\mathcal{F}$ is precipitous and the ultrapower is closed under λ sequences in $V[\mathcal{G}]$. For more on the basic facts about generic embeddings, see [For86].

The club filter on $\mathcal{P}(\lambda)$ ($\mathrm{CF}_{\mathcal{P}(\lambda)}$ or just CF) consists of all $A \subseteq \mathcal{P}(\lambda)$ such that $\exists f\colon \lambda^{<\omega} \to \lambda$ with $\mathrm{cl}_f \subseteq A$ ($\mathrm{cl}_f = \{a \subseteq \lambda \mid f''a^{<\omega} \subseteq a\}$). Sets in CF^+ are called stationary. CF is the smallest normal filter on $\mathcal{P}(\lambda)$.

If $S \in \mathcal{F}^+$, then $\mathcal{F} \restriction S =_{\mathrm{def}} \{A \subseteq \mathcal{P}(\lambda) \mid (\exists C \in \mathcal{F})\ C \cap S \subseteq A\}$ is a normal filter. If $S \in \mathrm{CF}^+$, then the club filter on S, $\mathrm{CF} \restriction S$, is the smallest normal filter on $\mathcal{P}(\lambda)$ containing S.

$\mathcal{P}_\kappa\lambda =_{\mathrm{def}} \{a \subseteq \lambda \mid |a| < \kappa \ \&\ a \cap \kappa \in \kappa\}$. This definition is slightly non-standard: usually the condition "$a \cap \kappa \in \kappa$" is dropped. The set $\mathcal{P}_\kappa\lambda$ is stationary in $\mathcal{P}(\lambda)$. If $\mathcal{F}$ is a filter on $\mathcal{P}(\lambda)$ and $\mathcal{P}_\kappa\lambda \in \mathcal{F}$, then $\mathcal{F}$ is κ-complete, and so $\forall s \in \mathcal{P}_\kappa\lambda$, $\{a \in \mathcal{P}_\kappa\lambda \mid s \subseteq a\} \in \mathcal{F}$.

If $a \subseteq \mathrm{Ord}$, then $\mathrm{cof}(a)$ is the cofinality of the order type of a. A $\diamond_{\kappa,\lambda}$ sequence is a set $\langle s_a \subseteq a\colon a \in \mathcal{P}_\kappa\lambda\rangle$ such that for all $A \subseteq \lambda$, $\{a \in \mathcal{P}_\kappa\lambda \mid a \cap A = s_a\}$ is stationary.

The following fact was proved in [BTW77] for filters on cardinals. A similar proof works here.

FACT 1.1: Assume $\mathcal{F}$ is a filter on $\mathcal{P}(\lambda)$. $\mathcal{F}$ is saturated iff for all filters $\mathcal{G} \supseteq \mathcal{F}$, $\exists S \in \mathcal{F}^+$ such that $\mathcal{G} = \mathcal{F} \restriction S$.

COROLLARY 1.2: *Suppose the club filter on S is saturated. Then every filter on S is saturated.*

2. Saturation

One of the first results about the failure of saturation is a theorem of Shelah ([She82], p. 440) that says, for example, if $\mathcal{F}$ is a saturated filter on ω_2, then $\{\alpha < \omega_2 \mid \mathrm{cof}(\alpha) = \omega_1\} \in \mathcal{F}$. The proof of this uses the following result (with $\lambda = \omega_2$). We also use this result to get similar facts about saturated filters on $\mathcal{P}_\kappa\lambda$.

THEOREM 2.1 ([She82], [Cum97]): *Assume $V \subseteq W$ are inner models of ZFC, λ is a cardinal of V, ρ is a cardinal of W, and $\lambda^+_V = \rho^+_W$. Assuming (*), $W \models \mathrm{cof}(\lambda) = \mathrm{cof}(\rho)$.*

(*) *λ is regular, or (λ is singular and) there is a good scale on λ, or (λ is singular and) W is a λ^+-cc forcing extension of V.*

See the next section for the definition of good scale.

Definition 2.2: $S_\lambda =_{\text{def}} \{a \subseteq \lambda |\ \text{cof}(a) = \text{cof}(|a|)\}$.

THEOREM 2.3: *Assume $\mathcal{F}$ is a saturated filter on $\mathcal{P}(\lambda)$. Then $S_\lambda \in \mathcal{F}$.*

Proof: Suppose not. So we get $j: V \to M \subseteq V[G]$ with $\mathcal{P}(\lambda) \setminus S_\lambda \in G$. Since $\mathcal{P}(\lambda) \setminus S_\lambda \in G$, $M \models \text{cof}(\lambda) \neq \text{cof}(|\lambda|)$. Since $M^\lambda \subseteq M$ in $V[G]$, $V[G] \models \text{cof}(\lambda) \neq \text{cof}(|\lambda|)$. This contradicts Theorem 2.1 since $V[G]$ is a λ^+-cc generic extension of V. ∎

LEMMA 2.4: *Assume $\kappa = \rho^+$, $\text{cof}(\lambda) < \kappa$, and $\text{cof}(\lambda) \neq \text{cof}(\rho)$. Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is non-stationary.*

Proof: Let $a \in \mathcal{P}_\kappa\lambda$. On a club, $|a| = \rho$ and so $\text{cof}(|a|) = \text{cof}(\rho)$. Since $\text{cof}(\lambda) < \kappa$, on a club $\sup(a) = \lambda$ and so $\text{cof}(a) = \text{cof}(\lambda)$. Therefore $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is non-stationary. ∎

COROLLARY 2.5: *For κ, λ as above, there is no saturated filter on $\mathcal{P}_\kappa\lambda$.*

Remark: If $\kappa = \rho^+$ and $\text{cof}(\lambda) = \text{cof}(\rho)$, then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$.

LEMMA 2.6: *Assume $\kappa = \rho^+ \geq \aleph_2$ and $\text{cof}(\lambda) \geq \kappa$. Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary, co-stationary in $\mathcal{P}_\kappa\lambda$.*

Proof: Let $f: \lambda^{<\omega} \to \lambda$. We may assume $a \in \mathcal{P}_\kappa\lambda \cap \text{cl}_f$ implies $\text{cof}(|a|) = \text{cof}(\rho)$. For any regular $\delta < \kappa$ we can build a continuous increasing chain of length δ to find $a \in \mathcal{P}_\kappa\lambda$ closed under f with $\text{cof}(a) = \delta$. Taking $\delta = \text{cof}(\rho)$ shows that $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary. Taking $\delta \neq \text{cof}(\rho)$ shows that $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$. ∎

COROLLARY 2.7: *For κ, λ as above, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Remark: If $\kappa = \aleph_1$, then for all $\lambda \geq \kappa$, $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$.

LEMMA 2.8: *Assume κ is a regular limit cardinal and $\text{cof}(\lambda) \neq \kappa$. Then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary, co-stationary in $\mathcal{P}_\kappa\lambda$.*

Proof: Let $f: \lambda^{<\omega} \to \lambda$ and $\rho < \kappa$ a regular cardinal. It is easy to find $a \in \mathcal{P}_\kappa\lambda \cap \text{cl}_f$ such that $|a| = |a \cap \kappa|$ and $\text{cof}(|a \cap \kappa|) = \rho$ and, if $\text{cof}(\lambda) > \kappa$, $\text{cof}(a) = \rho$ (if $\text{cof}(\lambda) < \kappa$, then for club many $a \in \mathcal{P}_\kappa\lambda$, $\text{cof}(a) = \text{cof}(\lambda)$). Hence

$S_\lambda \cap \mathcal{P}_\kappa\lambda$ is stationary (take $\rho = \mathrm{cof}(\lambda)$ if $\mathrm{cof}(\lambda) < \kappa$). If $\mathrm{cof}(\lambda) < \kappa$ then $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$—take $\rho \neq \mathrm{cof}(\lambda)$. Finally, assume $\mathrm{cof}(\lambda) > \kappa$. The idea for the following argument is from [Bau91]. Let $\delta = \mathrm{cof}(\lambda)$. Note that $\{a \in \mathcal{P}_\kappa\lambda \mid \mathrm{cof}(a) = \mathrm{cof}(a \cap \delta)\}$ is club, so we may assume f witnesses this. Let $\bar{f}\colon \delta^{<\omega} \to \delta$ be such that $a \in \mathrm{cl}_{\bar{f}}$ implies $\mathrm{cl}_f(a) \cap \delta = a$. Define $g\colon \delta \to \delta$ by $g(\alpha) = \sup(\mathrm{cl}_{\bar{f}}(\alpha + 1))$. Now choose $a \in \mathcal{P}_\kappa\delta$ such that $a \in \mathrm{cl}_{\bar{f}}$, $a \in \mathrm{cl}_g$, $|a| = |a \cap \kappa|$, $\mathrm{cof}(|a \cap \kappa|) = \aleph_1$, $\mathrm{cof}(a) = \aleph_1$, and $\kappa \in a$. Let $a_0 = a \cap \kappa$. Given a_n, let $\beta \in a \smallsetminus \sup(a_n)$, and $a_{n+1} = \mathrm{cl}_{\bar{f}}(a_n \cup \{\beta\})$. Let $a_\omega = \bigcup_{n\in\omega} a_n$. Then $a_\omega \cap \kappa = a \cap \kappa$, $a_\omega \in \mathrm{cl}_{\bar{f}}$ and $\mathrm{cof}(a_\omega) = \omega$. Let $b = \mathrm{cl}_f(a_\omega)$. Then $b \in \mathrm{cl}_f$ and $\mathrm{cof}(|b|) = \aleph_1$ and $\mathrm{cof}(b) = \omega$. Hence $S_\lambda \cap \mathcal{P}_\kappa\lambda$ is co-stationary in $\mathcal{P}_\kappa\lambda$. ∎

COROLLARY 2.9: *For κ, λ as above, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Remark: Assume $\mathrm{cof}(\lambda) = \kappa$ (κ regular limit). Then for club many $a \in \mathcal{P}_\kappa\lambda$, $\mathrm{cof}(a) = \mathrm{cof}(a \cap \kappa)$. So S_λ is club in the (stationary) set

$$\{a \in \mathcal{P}_\kappa\lambda \mid |a| = |a \cap \kappa|\}$$

and is non-stationary in the (possibly non-stationary) set

$$\{a \in \mathcal{P}_\kappa\lambda \mid |a| = |a \cap \kappa|^+\}.$$

The above method does not handle the cases: (i) $\kappa = \aleph_1$, (ii) $\kappa = \rho^+$ and $\mathrm{cof}(\lambda) = \mathrm{cof}(\rho)$, and (iii) κ regular limit and $\mathrm{cof}(\lambda) = \kappa$. Case (ii) is handled in the following:

THEOREM 2.10: *Assume $\mathrm{cof}(\lambda) < \kappa$ and $\kappa \geq \aleph_2$. Then the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Proof: Let $\langle f_\alpha : \alpha \in \lambda^+\rangle$ be a scale on λ (see Definition 3.3, so each $f_\alpha \in \Pi_{\xi<\mathrm{cof}(\lambda)}\rho_\xi$, where the ρ_ξ's are an increasing sequence of regular cardinals cofinal in λ with $\kappa < \rho_0$). Given $a \in \mathcal{P}_\kappa\lambda$ define $g_a \in \Pi\rho_\xi$ by $g_a(\xi) = \sup(a \cap \rho_\xi)$ and let $\pi(a) =$ least $\alpha \in \lambda^+$ such that $g_a \leq^* f_\alpha$. Let $\mathcal{F}$ be a filter on $\mathcal{P}_\kappa\lambda$. Let $\theta >> \lambda$, and assume $\mathcal{F}_\theta$ is a filter on $\mathcal{P}_\kappa H_\theta$ projecting to $\mathcal{F}$. Let

$$E = \{b \prec H_\theta \mid b \in \mathcal{P}_\kappa H_\theta \ \&\ \langle f_\alpha : \alpha \in \lambda^+\rangle \in b \ \&\ \mathrm{cof}(\lambda) \subseteq b \ \&\ \langle \rho_\xi : \xi < \mathrm{cof}(\lambda)\rangle \in b\}.$$

CLAIM 1: *If $b \in E$ then $\sup(b \cap \lambda^+) \leq \pi(b \cap \lambda)$.*

Suppose not. Let $b \in E$ with $\sup(b\cap\lambda^+) > \pi(b\cap\lambda) =_{\mathrm{def}} \xi$. Say $\beta \in b\cap\lambda^+$ with $\beta > \xi$. But $f_\beta \mathrel{{}^*{>}} f_\xi \mathrel{{}^*{\geq}} g_{b\cap\lambda}$. Therefore $\exists\eta < \mathrm{cof}(\lambda)$ such that $f_\beta(\eta) > f_\alpha(\eta) \geq$

$g_{b\cap\lambda}(\eta)$. But $\mathrm{cof}(\lambda) \subseteq b$, $\beta \in b$, and $\langle f_\alpha : \alpha \in \lambda^+\rangle \in b$. Therefore $f_\beta(\eta) \in b \cap \rho_\eta$. But $g_{b\cap\lambda}(\eta) = \sup(b \cap \rho_\eta)$. Contradiction

Let $T \in \mathcal{F}_\theta$ and define $S_T(a) = \sup\{\sup(b\cap\lambda^+)|\ b\cap\lambda = a \ \&\ b \in T\}$. Note that S_T is defined on a set in $\mathcal{F}$ (the projection of T), if $T \subseteq T'$ then $S_T(a) \leq S_{T'}(a)$, and if $T \subseteq E$ then $S_T(a) \leq \pi(a)$.

CLAIM 2: *Given $\beta \in \lambda^+$ and $T \subseteq E$ with $T \in \mathcal{F}_\theta$, on an $\mathcal{F}$ measure one set we have $\beta < S_T(a) \leq \pi(a) < \lambda^+$.*

We already have that S_T is defined on an $\mathcal{F}$ measure one set and $S_T(a) \leq \pi(a) < \lambda^+$. Let $\beta \in \lambda^+$ and let $T' = \{b \in T|\ \beta \in b\}$. Then $T' \in \mathcal{F}_\theta$ and (on the projection of T') $S_{T'}(a) > \beta$ and $S_{T'}(a) \leq S_T(a)$.

CLAIM 3: *Assume $f\colon \mathcal{P}_\kappa\lambda \to \lambda^+$ is such that $\Vdash_{\mathcal{F}^+}[f] = \sup j''\lambda^+$. Then there is a $T \in \mathcal{F}_\theta$ such that $(\forall T' \subseteq T)\ T' \in \mathcal{F}_\theta$ on a set in $\mathcal{F}$, $S_{T'}(a) = f(a)$.*

On an $\mathcal{F}_\theta$ measure one set $f(b\cap\lambda) \geq \sup(b\cap\lambda^+)$ (If not, then there is a $S \in \mathcal{F}_\theta^+$ such that $f(b \cap \lambda) < \sup(b \cap \lambda^+)$, so on some $S' \in \mathcal{F}_\theta^+$, $f(b \cap \lambda) < \eta$ ($\eta \in \lambda^+$ is fixed). Projecting to $\mathcal{F}$ we get $\bar{S} \in \mathcal{F}^+$ such that $f(b) < \eta$ on $\bar{S}$. Contradiction.)

So let $T \in \mathcal{F}_\theta$ such that $T \subseteq E$ and $(\forall b \in T)\ f(b \cap \lambda) \geq \sup(b \cap \lambda^+)$. Therefore, on an $\mathcal{F}$ measure one set, $f(a) \geq S_T(a)$. Suppose on $S \in \mathcal{F}^+$ we have $f(a) > S_T(a)$. Then since $\Vdash[f] = \sup j''\lambda^+$, there exists $\bar{S} \subseteq S$ and $\eta < \lambda^+$ such that on $\bar{S}$, $S_T(a) \leq \eta$. This contradicts Claim 2. Finally, assume $T' \subseteq T$. Then on an $\mathcal{F}$ measure one set $S_{T'}(a) \leq S_T(a) = f(a)$. Again by Claim 2, $S_{T'}(a) = f(a)$ on an $\mathcal{F}$ measure one set.

CLAIM 4: *Assume $\rho < \kappa$ is regular, $\rho \neq \mathrm{cof}(\lambda)$, $T \subseteq \mathcal{P}_\kappa H_\theta$ is stationary, $T \subseteq E$, and $\forall a \in T$, a is IA (internally approachable) of length ρ (this means there is an increasing, continuous sequence $\langle a_\xi : \xi < \rho\rangle$ where each $a_\xi \in E$, $\forall \rho' < \rho\ \langle a_\xi : \xi < \rho'\rangle \in a$, and $a = \bigcup_{\xi<\rho} a_\xi$ — see [FMS88]). Let $\bar{T}$ be the projection of T to $\mathcal{P}_\kappa\lambda$. Then for all $a \in \bar{T}$, $S_T(a) = \pi(a)$ and $\mathrm{cof}(\pi(a)) = \rho$.*

The idea for the proof of Claim 4 comes from [FM97]. Let $b \in T$, and $\langle b_\alpha : \alpha < \rho\rangle$ be a witness to IA of length ρ. We may assume $(\forall\alpha \in \rho)\ b_\alpha \in b_{\alpha+1}$. Let $a = b \cap \lambda$. It is enough to see that $\sup(b \cap \lambda^+) = \pi(a)$. (Note that $\mathrm{cof}(\sup(b \cap \lambda^+)) = \rho$.) Given $\alpha < \rho$ we have $(\forall f \in b_\alpha)\ f < g_{b_\alpha}$ (everywhere) and, since $g_{b_\alpha} \in b_{\alpha+1}$, there is $\gamma_\alpha \in b_{\alpha+1}$ such that $g_{b_\alpha} \leq^* f_{\gamma_\alpha}$. By Claim 1, $\pi(a) \geq \sup(b \cap \lambda^+)$. So let $\delta = \sup(b \cap \lambda^+)$ and we will show $g_b \leq^* f_\delta$. For all $\alpha < \rho$, $g_{b_\alpha} \leq^* f_{\gamma_\alpha} <^* f_\delta$. Since $\rho \neq \mathrm{cof}(\lambda)$, $\exists A \subseteq \rho$ unbounded and $\nu < \mathrm{cof}(\lambda)$ such that $\forall\alpha \in A$ and $\forall\xi \in (\nu, \mathrm{cof}(\lambda))\ g_{b_\alpha}(\xi) < f_\delta(\xi)$. But $g_b(\xi) = \sup_{\alpha\in A} g_{b_\alpha}(\xi)$ and so $g_b(\xi) \leq f_\delta(\xi)$.

Let $\rho < \kappa$ be regular, $\rho \neq \mathrm{cof}(\lambda)$. Let $T = \{b \in E |\ b$ is IA of length $\rho\}$.

CLAIM 5: *T is stationary.*

Let $f\colon H_\theta^{<\omega} \to H_\theta$. Let $b_0 \in E \cap \mathrm{cl}_f$. If $\xi < \rho$ is limit let $b_\xi = \cup_{\varepsilon<\xi} b_\varepsilon$. Given b_ξ, let $b_{\xi+1} \in E \cap \mathrm{cl}_f$ such that $b_\xi \cup \{\langle b_\varepsilon : \varepsilon \leq \xi\rangle\} \in b_{\xi+1}$. So $b = \bigcup_{\varepsilon<\rho} b_\varepsilon \in E \cap \mathrm{cl}_f$. To see b is IA of length ρ we just need $\forall \xi < \rho\ \langle b_\alpha : \alpha \in \xi\rangle \in b$. But $\langle b_\alpha : \alpha \in \xi\rangle \in b_{\xi+1} \subseteq b$.

Finally, let $\mathcal{F}_\theta = \mathrm{CF} \restriction T$. $\mathcal{F}$ is gotten by projection. We will show that $\mathcal{F}$ is not saturated, and therefore by Corollary 1.2 the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.

For a contradiction, assume $\mathcal{F}$ is saturated. So there is an $f\colon \mathcal{P}_\kappa\lambda \to \lambda^+$ such that $\Vdash [f] = \sup j''\lambda^+$, and on a set in $\mathcal{F}$, $\mathrm{cof}(f(a)) > \rho$ (otherwise we could force to have $\mathrm{cof}([f]) \leq \rho$ in the ultrapower—so this collapses λ^+).

By Claim 3, $\exists R \in \mathcal{F}_\theta$ such that for any $R' \subseteq R$ ($R' \in \mathcal{F}_\theta$) on a set in $\mathcal{F}$, $S_{R'}(a) = f(a)$. So on a set in $\mathcal{F}$, $S_{R\cap T}(a) = f(a)$. But $R \cap T$ is a set as in Claim 4. Hence on a set in $\mathcal{F}$ (the projection of $R \cap T$) $S_{R\cap T}(a) = \pi(a)$ and $\mathrm{cof}(\pi(a)) = \rho$. Therefore on a set in $\mathcal{F}$, $\mathrm{cof}(f(a)) = \rho$. This contradiction completes the proof. ∎

QUESTION: *In the above proof, $\mathcal{F}$ is the projection of* CF $\restriction T$. *Is $\mathcal{F}$ the club filter restricted to a stationary set?*

We conclude this section with three previously known theorems.

THEOREM 2.11 ([GS97]): *For all $\kappa > \aleph_1$, the club filter on κ is not saturated. In fact, for any regular ρ with $\rho^+ < \kappa$,* CF $\restriction \{\alpha < \kappa |\ \mathrm{cof}(\alpha) = \rho\}$ *is not saturated.*

COROLLARY 2.12: *For all regular κ and all regular $\lambda \geq \aleph_2$, the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

Proof: Define $g\colon \mathcal{P}_\kappa\lambda \to \lambda$ by $g(a) = \sup(a)$. Suppose $S \subseteq \lambda$ is stationary and $(\forall\alpha \in S)\ \mathrm{cof}(\alpha) < \kappa$. Then $g^{-1}(S)$ is stationary (let $f\colon \lambda^{<\omega} \to \lambda$ and choose $\alpha \in S$ such that α is closed under f. Now build $a \in \mathcal{P}_\kappa\lambda \cap \mathrm{cl}_f$ such that $\sup(a) = \alpha$). Also, if $S \subseteq \mathcal{P}_\kappa\lambda$ is stationary, then $g''S \subseteq \lambda$ is stationary (if $f\colon \lambda^{<\omega} \to \lambda$, define $h(\alpha) = \mathrm{cl}_f(\alpha+1)$). If $a \in S \cap \mathrm{cl}_h$, then $\sup(a)$ is closed under f. The result now follows from Theorem [2.11]. ∎

THEOREM 2.13 ([DM93]): *If $\lambda > 2^{<\kappa}$ then $\diamond_{\kappa,\lambda}$ holds. Hence the club filter on $\mathcal{P}_\kappa\lambda$ is not saturated.*

THEOREM 2.14 ([BT82]): *For any $\lambda > \aleph_1$, $\mathcal{P}_{\aleph_1}\lambda$ can be split into 2^ω many disjoint stationary sets.*

Remark: Piecing everything together, we have the following partial results towards the theorem of Foreman and Magidor: The club filter on $\mathcal{P}_\kappa\lambda$ is not saturated unless

1. $\kappa = \lambda = \aleph_1$ (consistent).
2. $\kappa = \aleph_1$, $\lambda = 2^\omega$ is singular.
3. κ is limit and $\mathrm{cof}(\lambda) = \kappa$ and $2^{<\kappa} \geq \lambda$.

3. Cardinal preserving to pre-saturation

A filter $\mathcal{F}$ on $\mathcal{P}(\lambda)$ is **weakly pre-saturated** if $\mathcal{F}$ is precipitous and $\Vdash_{\mathcal{F}^+}$ " λ^+ is preserved". The filter $\mathcal{F}$ is called **cardinal preserving** if $\Vdash_{\mathcal{F}^+}$ " λ^+ is preserved". If $|\mathcal{F}^+| = \lambda^+$, then pre-saturated, weakly pre-saturated and cardinal preserving are all equivalent. It is not known if they are equivalent in general.

We use a number of known combinatorial principles to get that the club filter cannot have these strong properties. For the case λ regular, the solution is complete—the club filter on $\mathcal{P}_\kappa\lambda$ is not cardinal preserving unless $\kappa = \aleph_1$ or $\kappa = \lambda$ is weakly inaccessible (and both these cases are consistent).

Definition 3.1: $\mathrm{Sh}(\lambda)$ means for any $\mathbb{P} \in V$, if $V^{\mathbb{P}} \models \mathrm{cof}(\lambda) \neq \mathrm{cof}(|\lambda|)$, then $V^{\mathbb{P}}$ collapses λ^+_V.

Definition 3.2: $\mathrm{AD}(\lambda)$ means $\exists \langle a_\alpha : \alpha \in \lambda^+\rangle$ such that each a_α is an unbounded subset of λ and $\forall \alpha \in \lambda^+ \exists f_\alpha$: $\alpha \to \lambda$ such that $\beta_1 < \beta_2 < \alpha$ implies $[a_{\beta_1} \smallsetminus f_\alpha(\beta_1)] \cap [a_{\beta_2} \smallsetminus f_\alpha(\beta_2)] = \emptyset$.

Definition 3.3: Suppose λ is singular. A *scale* on λ is an increasing sequence of regular cardinal $\langle \rho_\xi : \xi \in \mathrm{cof}(\lambda)\rangle$ cofinal in λ, and a sequence $\langle f_\alpha : \alpha \in \lambda^+\rangle$ such that for each α, $f_\alpha \in \Pi_{\xi\in\mathrm{cof}(\lambda)}\rho_\xi$, $\alpha < \alpha'$ implies $f_\alpha <^* f_{\alpha'}$, and $\forall f \in \Pi_{\xi\in\mathrm{cof}(\lambda)}\rho_\xi$ $\exists \alpha \in \lambda^+$ such that $f <^* f_\alpha$. We will assume $\langle \rho_\alpha : \alpha \in \mathrm{cof}(\lambda)\rangle$ is discontinuous everywhere and $\forall \alpha \in \lambda^+$ $\forall \xi \in \mathrm{cof}(\lambda)$ $f_\alpha(\xi) > \sup\{\rho_{\xi'} \mid \xi' < \xi\}$. An ordinal γ is **good** for $\langle f_\alpha : \alpha \in \lambda^+\rangle$ if $\exists A \subseteq \gamma$ unbounded and $\sigma < \mathrm{cof}(\lambda)$ such that $\forall \alpha < \alpha'$ from A and $\nu \in (\sigma, \mathrm{cof}(\lambda))$ $f_\alpha(\nu) < f_{\alpha'}(\nu)$. The scale is **good** if $\exists$ club $C \subseteq \lambda^+$ such that $\forall \alpha \in C$ if $\mathrm{cof}(\alpha) > \mathrm{cof}(\lambda)$, then α is good for the scale. $\mathrm{GS}(\lambda)$ means there is a good scale on λ.

Remarks:

1. λ regular implies $\mathrm{AD}(\lambda)$.

2. $\mathrm{AD}(\lambda)$ implies $\mathrm{Sh}(\lambda)$. [She82]
3. $\mathrm{GS}(\lambda)$ and λ singular implies $\mathrm{Sh}(\lambda)$. [Cum97]
4. $\square^*_\lambda$ implies $\mathrm{AD}(\lambda)$. [CFM]
5. It is not known if $\exists\lambda\neg\mathrm{Sh}(\lambda)$ is consistent (it is consistent to have $\exists\lambda[\neg\,\mathrm{AD}(\lambda)$ and $\neg\,\mathrm{GS}(\lambda)])$.
6. Shelah has proved that there is a scale for all singular λ and that the set of good points is stationary for all scales ([HJS86]; also see [Cum97]). Shelah also gives an example of a model with no good scale ([HJS86]). Another example of a model with no good scale is given by Foreman and Magidor in [FM97], where they show a version of Chang's Conjecture, $(\aleph_{\omega+1},\aleph_\omega)\twoheadrightarrow(\aleph_1,\aleph_0)$, implies there is no good scale on $\aleph_\omega$.

The proof of the following theorem is essentially the same as Theorem 2.3.

THEOREM 3.4: *Assume* $\mathrm{Sh}(\lambda)$ *and* $\mathcal{F}$ *is a pre-saturated filter on* $\mathcal{P}(\lambda)$. *Then* $S_\lambda\in\mathcal{F}$.

THEOREM 3.5: *Suppose* $\mathcal{F}$ *is a cardinal preserving filter on* $\mathcal{P}(\lambda)$ *and* $\mathrm{AD}(\lambda)$. *Then* $S_\lambda\in\mathcal{F}$.

Proof: We will use Shelah's method of proof of Theorem 2.1 (page 440, [She82]). Let $\langle a_\alpha:\alpha\in\lambda^+\rangle$, $\langle f_\alpha:\alpha\in\lambda^+\rangle$ witness $\mathrm{AD}(\lambda)$. Suppose $S_\lambda\notin\mathcal{F}$. Let $G\subseteq\mathcal{F}^+$ be generic with $\mathcal{P}(\lambda)\setminus S_\lambda\in G$. So we get $j\colon V\to(M,E)\subseteq V[G]$ with $\lambda^+\subseteq M$ (we collapse the well-founded part of M), and $\mathcal{P}^V(\lambda)\subseteq M$, and $M\models\mathrm{cof}(\lambda)\neq\mathrm{cof}(|\lambda|)$. Work in M: we write $\lambda=\bigcup_{\alpha\in\mathrm{cof}(|\lambda|)}A_\alpha$ where the A_α's are increasing, continuous and $|A_\alpha|<|\lambda|$. So if $a\subseteq\lambda$ is unbounded, then $\exists\alpha<\mathrm{cof}(|\lambda|)$ such that $a\cap A_\alpha$ is unbounded in λ. Now work in $V[G]$: we have $\forall\alpha\in\lambda^+\,\exists\beta\in\mathrm{cof}(|\lambda|)^M$ such that $a_\alpha\cap A_\beta$ is unbounded in λ. So there is a fixed β_0 and an unbounded $\mathcal{A}\subseteq\lambda^+$ such that $(\forall\alpha\in\mathcal{A})\ a_\alpha\cap A_{\beta_0}$ is unbounded in λ. Let $\alpha_0\in\mathcal{A}$ be such that $\mathcal{A}\cap\alpha_0$ has order type λ. Note that $\langle a_\alpha:\alpha\in\alpha_0\rangle, A_{\beta_0}$, and f_{α_0} are all in M. Now work in M: The set

$$\{(a_\alpha\cap A_{\beta_0})\setminus f_{\alpha_0}(\alpha)|\ \alpha<\alpha_0\ \&\ a_\alpha\cap A_{\beta_0}\text{ is unbounded in }\lambda\}$$

is a family of $|\lambda|$ many non-empty pairwise disjoint subsets of A_{β_0}. But $|A_{\beta_0}|<|\lambda|$, contradiction. ∎

As in section 2, these two theorems have the following three corollaries:

COROLLARY 3.6: *Assume* $\mathrm{AD}(\lambda)$ *[*$\mathrm{Sh}(\lambda)$*],* $\kappa=\rho^+$, $\mathrm{cof}(\lambda)<\kappa$, *and* $\mathrm{cof}(\lambda)\neq\mathrm{cof}(\rho)$. *Then there is no cardinal preserving [pre-saturated] filter on* $\mathcal{P}_\kappa\lambda$.

COROLLARY 3.7: *Assume* AD(λ) *[*Sh(λ)*]*, $\kappa = \rho^+ \geq \aleph_2$ *and* $\mathrm{cof}(\lambda) \geq \kappa$. *Then the club filter on* $\mathcal{P}_\kappa\lambda$ *is not cardinal preserving [pre-saturated].*

COROLLARY 3.8: *Assume* AD(λ) *[*Sh(λ)*]*, κ *is a regular limit cardinal and* $\mathrm{cof}(\lambda) \neq \kappa$. *Then the club filter on* $\mathcal{P}_\kappa\lambda$ *is not cardinal preserving [pre-saturated].*

THEOREM 3.9: *Assume* $\mathrm{cof}(\lambda) < \kappa$ *and there is a good scale on* λ. *Then there is no weakly pre-saturated filter on* $\mathcal{P}_\kappa\lambda$.

Proof: Suppose not. So there is $j\colon V \to M \subseteq V[G]$ such that λ_V^+ is still a cardinal of $V[G]$, M is well-founded, $\mathcal{P}^V(\lambda) \subseteq M$, and $\mathrm{cp}(j) = \kappa$ with $j(\kappa) > \lambda$. Let $\langle f_\alpha : \alpha \in \lambda^+\rangle$ be a good scale on λ. So there is a club $C \subseteq \lambda^+$ such that $\alpha \in C$ and $\mathrm{cof}(\alpha) > \mathrm{cof}(\lambda)$ implies α is good for $\langle f_\alpha : \alpha \in \lambda^+\rangle$. Let $\rho = \sup j''\lambda^+$ Note that $\rho < j(\lambda^+)$ (see [BM97]) and so $\rho \in j(C)$. Since $V[G] \models \mathrm{cof}(\rho) = \lambda^+$, $M \models \mathrm{cof}(\rho) \geq \lambda^+ > \mathrm{cof}(\lambda)$. So in M, there is an $A \subseteq \rho$ such that $\sup(A) = \rho$ and $\exists\sigma < \mathrm{cof}(\lambda)$ such that $\alpha_1 < \alpha_2$ from A and $\nu \in (\sigma, \mathrm{cof}(\lambda))$ implies $j(f)_{\alpha_1}(\nu) < j(f)_{\alpha_2}(\nu)$. Now work in $V[G]$ and repeat an argument from [Cum97]. For each α in λ^+ choose $\beta_\alpha < \delta_\alpha$ from A and $\gamma_\alpha \in \lambda^+$ such that $\beta_\alpha < j(\gamma_\alpha) < \delta_\alpha$. Do this so $\alpha_1 < \alpha_2$ implies $\delta_{\alpha_1} < \beta_{\alpha_2}$ and $\sup\{\beta_\alpha \mid \alpha \in \lambda^+\} = \rho$. For each $\alpha \in \lambda^+$ $\exists\sigma_\alpha < \mathrm{cof}(\lambda)$ such that $j(f)_{\beta_\alpha} < j(f)_{j(\gamma_\alpha)} < j(f)_{\delta_\alpha}$ beyond σ_α. Since λ^+ is regular there is an unbounded $B \subseteq \lambda^+$ and fixed σ_1 such that $\forall\alpha \in B$ $\sigma_\alpha = \sigma_1$. Let $\bar\sigma = \max(\sigma, \sigma_1)$. But then if $\alpha_1 < \alpha_2$ are from B, then $f_{\gamma_{\alpha_1}}(\bar\sigma + 1) < f_{\gamma_{\alpha_2}}(\bar\sigma + 1)$. Hence λ^+ must be collapsed in $V[G]$. ∎

Precipitousness is ruled out under certain conditions by the following theorem of Matsubara and Shioya.

THEOREM 3.10 ([MS]): *If* $\lambda^{<\kappa} = 2^\lambda$ *and* $2^{<\kappa} < 2^\lambda$, *then the club filter on* $\mathcal{P}_\kappa\lambda$ *is nowhere precipitous.*

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